Geometry and Topology
For morning of October 24th

Problem 1 *Let Ω be a domain in $\mathbb{R}^n$, containing the ball D_r of radius r with center at the origin. Consider $u : \Omega \to \mathbb{R}$ satisfying the minimal graph equation:*

$$div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Namely, $\Gamma_u = \{(x, u(x))\,|x \in \Omega\}$ is a minimal graph in $\mathbb{R}^{n+1}$. Let $B_r \subset \mathbb{R}^{n+1}$ be the ball of radius r centered at the origin. Show that

$$Vol\left(B_r \cap \Gamma_u\right) \leq \frac{Vol\left(S^n\right)}{2} r^n$$

where S^n is the unit hypersphere of $\mathbb{R}^{n+1}$.

Problem 2 *Show that $\int_\Sigma H^2 \geq 16\pi$ for any closed embedded surface Σ in $\mathbb{R}^3$. When does equality hold? (Here H is the mean curvature of the surface Σ, namely H is the sum of principal curvatures).*

Problem 3 *Let I be the interval $[0,1]$. For a topological space B, say homeomorphisms $g_0, g_1 : B \to B$ are isotopic if they are homotopic via a homotopy $G : B \times I \to B \times I$ with each $G_t : B \to B$ defined by $G_t(b) = G(b,t)$ also a homeomorphism.*

(a) Show that any order-preserving homeomorphism $f : I \to I$ is isotopic to the identity.

(b) Show that any homeomorphism $f : S^1 \to S^1$ of the unit circle is isotopic to the identity or the reflection along the x-axis.